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1987 J. Phys. A: Math. Gen. 20 35

(http://iopscience.iop.org/0305-4470/20/1/013)

J. Phys. A: Math. Gen. **20** (1987) 35-45. Printed in the UK

Some coupling and recoupling coefficients for symmetric representations of SO_n

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Received 21 August 1985, in final form 7 April 1986

Abstract. A new expression of $SO_n \supset SO_{n-1}$ isoscalar factors (reduced Wigner coefficients) for coupling $l_1 \times l_2$ to $[L_1 L_2]$ is obtained. Special classes of these factors (including those used together with special isofactors of $SU_n \supset SO_n$ in O_n-scalar basis functions depending on the microscopic collective variables of a nucleus) are shown as being proportional to the $6j$-coefficients of SU_2 with standard or quartervalued parameters. Group chain reflection symmetry of isofactors is discovered which allows us to interchange certain reflected parameters of irreducible representations of the group and its subgroup. Recoupling coefficients ($6l$-coefficients) for symmetric representations of SO_n ($n \geq 5$) are expanded in terms of $6j$-coefficients of SU_2 with standard or quartervalued parameters. New expressions for special Clebsch-Gordan and $6j$-coefficients of $SU_2(SO_3)$ are presented.

1. Introduction

In the explicit expressions of $U_{3n} \supset U_3 \times U_n \supset O_n$ basis functions depending on the microscopic collective variables of the nucleus (Vanagas and Kalinauskas 1974, Vanagas 1977, 1980), special isofactors (reduced Wigner coefficients) of $SU_n \supset SO_{n-1}$ are needed (together with those of $SU_n \supset SO_n$). The isofactors of $SU_n \supset SO_n$ for O_n-scalar basis functions (as well as for a closed shell case) were found by Ališauskas (1986a) (for more detailed references see Ališauskas 1986a). For the most general collective basis functions in a two-dimensional case (which may be used as a recurrent step for a three-dimensional case, see Vanagas and Katkevičius (1983)) special isofactors of $SU_n \supset SO_n$ were found by Ališauskas (1984).

In both two- and three-dimensional cases, isofactors of $SO_n \supset SO_{n-1}$,

$$\begin{bmatrix} l_1 & l_2 & [L_1 L_2]_n \\ l_1' & l_2' & [L_1' L_2']_{n-1} \end{bmatrix} \tag{1.1}$$

for coupling the states of symmetric irreducible representations (irreps) l_1 and l_2, with particular restrictions on irreps of the subgroup, are needed. Isofactors with all three irreps symmetric (i.e. with $L_2 = L_2' = 0$) were considered by Gavrilik (1973) and Kildyushov and Kuznetsov (1973), but only Norvaišas and Ališauskas (1974a) observed that such isofactors are the analytical continuation of semistretched isofactors of $Sp_4 \supset SU_2 \times SU_2$ of the second kind, introduced by Ališauskas and Jucys (1971). The substitution group technique of $Sp_4(SO_5)$ (Ališauskas and Jucys 1969) allowed us to derive expressions (Norvaišas and Ališauskas 1974a, see also (35) of Ališauskas 1983) for special isofactors of $SO_n \supset SO_{n-1}$ and $SO_n \supset SO_{n'} + SO_{n-n'}$ with $L_2 = 0$, which are considerably more convenient for the cases close to the stretched ones (i.e. with small values of $l_1 + l_2 - L_1$).

0305-4470/87/010035+11\$02.50

The relations of analytical continuation between the isofactors of $SO_n \supset SO_{n-1}$ (1.1) (and, more generally, of $SO_n \supset SO_{n'} + SO_{n-n'}$) and the isofactors of $Sp_4 \supset SU_2 \times SU_2$,

$$\begin{bmatrix} \langle K_1\Lambda_1\rangle & \langle K_2\Lambda_2\rangle & \langle K\Lambda\rangle^u \\ I_1J_1 & I_2J_2 & IJ \end{bmatrix} \tag{1.2}$$

with the parameters

$$\begin{aligned} &K_1 = -\tfrac{1}{4}(2L_2'+n-1) && K_2 = -\tfrac{1}{4} && K = -\tfrac{1}{4}(2L_2+n) \\ &\Lambda_1 = -\tfrac{1}{4}(2L_1'+n-1) && \Lambda_2 = -\tfrac{1}{4}(2\delta+1) && \Lambda = -\tfrac{1}{4}(2L_1+n) \\ &I_1 = -\tfrac{1}{4}(2l_1'+n-1) && I_2 = -\tfrac{1}{4}(2\delta_1+1) && I = -\tfrac{1}{4}(2l_1+n) \\ &J_1 = -\tfrac{1}{4}(2l_2'+n-1) && J_2 = -\tfrac{1}{4}(2\delta_2+1) && J = -\tfrac{1}{4}(2l_2+n) \end{aligned} \tag{1.3}$$

($\delta_1, \delta_2, \delta = 0$ or 1 so that $l_1 - l_1' - \delta_1$, $l_2 - l_2' - \delta_2$ and $L_1 + L_2 - L_1' - L_2' - \delta$ are even integers), as well as

$$\begin{bmatrix} l_1 & l_2 & [L_1L_2]_n \\ l_1' & l_2' & [L_1'L_2']_{n-1} \end{bmatrix} = \begin{bmatrix} l_1+\frac{1}{2}(n-5) & l_2+\frac{1}{2}(n-5) & [L_1+\frac{1}{2}(n-5), L_2+\frac{1}{2}(n-5)]_5 \\ l_1'+\frac{1}{2}(n-5) & l_2'+\frac{1}{2}(n-5) & [L_1'+\frac{1}{2}(n-5), L_2'+\frac{1}{2}(n-5)]_4 \end{bmatrix} \tag{1.4}$$

allowed us (Norvaišas and Ališauskas 1974b) to apply the results and techniques developed for $Sp_4 \supset SU_2 \times SU_2(SO_5 \supset SO_4)$ by Ališauskas and Jucys (1969, 1971) in the consideration of the more general isofactors (1.1). (Relations (1.3) and (1.4) were proved later by Ališauskas (1974, 1976, 1983) on the grounds of the complementarity of the groups SO_n and $Sp(4, R)$.)

In particular, relation (1.4) applied to different forms and variants of semistretched isofactors of Sp_4 (derived by the substitution group technique of Ališauskas and Jucys (1969)) allowed the expression of the isofactors (1.1) with parameters $l_1 + l_2 = L_1 + L_2$ or $l_1 - l_2 = L_1 - L_2$ as double sums (see (3.5) and (3.7) of Norvaišas and Ališauskas (1974b)). Similarly, relations (1.3) allowed us to express the isofactors (1.1) with parameters $L_2' = L_2$ or $L_1' = L_1$ as double sums (see (4.1) and (4.4) of Norvaišas and Ališauskas (1974b)). So all the above-mentioned cases of isofactors (1.1) appeared to be the incarnations of some more universal function, being of importance in the Wigner-Racah calculus of different Lie groups (see Ališauskas 1983) and generally represented as a triple sum.

The weight-lowering operators of Sp_4 (Ališauskas and Jucys 1971), together with (1.4), allows the expansion of the general isofactors (1.1) in terms of the above-mentioned particular cases† and to express them as sixfold sums. Expression (3.9) of Ališauskas and Norvaišas (1975), used together with (1.4), may be convenient for the cases of (1.1) close to the stretched ones. All the above-mentioned expressions simplify themselves with a decrease of the differences $l_1 - l_1'$, $l_2 - l_2'$, $L_1 - L_1'$ and $L_2 - L_2'$.

However, they remain complicated for small values of, e.g. l_2' or L_1', and do not satisfy the criterion of the polynomial representability (cf Castilho-Alcarás and Vanagas 1984, 1985) of the special isofactors of $SO_n \supset SO_{n-1}$ needed in the microscopic theory of collective nuclear states.

In this paper, the following special isofactors of $SO_n \supset SO_{n-1}$ for coupling of two symmetric irreps are shown to be proportional to the 6j-coefficients of SU_2 with standard

† Unfortunately, both corresponding equations (5.1) and (5.4) of Norvaišas and Ališauskas (1974b) contain several errata which are corrected in Ališauskas (1986b).

or quartervalued parameters†: (i) with $l'_1=L'_1$ and $l'_2=L'_2=0$; (ii) with $l'_1=l'_2$ and $L'_1=L'_2=L_2$, and (iii) with $l_1=l_2=l$ and $L_1=L_2=L'_1$. The isofactors of the two last classes appear to be interrelated by a special procedure of analytical continuation which allows us to interchange (after definite reflections) the parameters of irreps of the group and its subgroup and may be generalised for a wider class of isofactors. A new expression for the general isofactors (1.1) is obtained, which is convenient for low values of the difference $L'_1-L'_2$.

The above-mentioned results, as well as SU_2 recoupling techniques and methods used for the construction of biorthogonal systems (Ališauskas 1978b), allow us to expand the SO_n recoupling coefficients ($6l$-symbols) for all symmetric irreps in terms of $6j$-coefficients of SU_2 (with standard or quartervalued parameters). Originally these recoupling coefficients were supposed to be used for expansion of special isofactors of $SU_n \supset SO_n$, in a more convenient form presented by Ališauskas (1986a). These expressions will find applications in nuclear theory.

2. Isofactors of $SO_n \supset SO_{n-1}$ with special restrictions for the irreps of the subgroup

Let us prove the expression for special isofactors of $SO_n \supset SO_{n-1}$ needed for the microscopic theory of the collective nuclear states:

$$\begin{bmatrix} l_1 & l_2 & [L_1L_2]_n \\ l' & 0 & l' \end{bmatrix} = (-1)^{(l_1+l_2+L_1-L_2)/2+l'+n}\bigg((n-3)(2L_1+n-2)(2L_2+n-4)(L_1-L_2+1)$$
$$\times(L_1+L_2+n-3)\frac{(L_1+n-4)!(L_2+n-5)!l'!l_2!}{2(l'+n-4)!(l_2+n-3)!(L_1+1)!L_2!}\bigg)^{1/2}$$
$$\times\begin{Bmatrix} \frac{1}{2}(L_1+L_2+n)-2 & \frac{1}{2}l_1+\frac{1}{4}n-1 & \frac{1}{2}l_2+\frac{1}{4}n-1 \\ \frac{1}{2}l_1+\frac{1}{4}n-1 & \frac{1}{2}(L_1-L_2) & l'+\frac{1}{2}n-2 \end{Bmatrix}. \tag{2.1}$$

Here $n>4$; the $n=3$ case differs in phase factor (cf (2.14) of Ališauskas 1978a‡); the correct dimensional factor $[(L_1+L_2+1)(L_1-L_2+1)/(l_2+1)]^{1/2}$ in the RHS of (2.1), for $n=4$, may be found in the framework of angular momentum theory.

In the RHS of (2.1) the usual $6j$-coefficient of SU_2 appears for n even. For arbitrary n it may be calculated by means of the following expression:

$$\begin{Bmatrix} \frac{1}{2}(L_1+L_2+n)-2 & \frac{1}{2}l_1+\frac{1}{4}n-1 & \frac{1}{2}l_2+\frac{1}{4}n-1 \\ \frac{1}{2}l_1+\frac{1}{4}n-1 & \frac{1}{2}(L_1-L_2) & l'+\frac{1}{2}n-2 \end{Bmatrix}$$
$$=(-1)^{(l_1+l_2+L_1-L_2)/2+l'+n}2^{L_2+n/2-1}\nabla_{n[0,4]}(l_1l_2;\,L_1L_2)$$
$$\times(2l'+n-4)!!\bigg(\frac{(L_1-l')!(l_1-l')!(l'-L_2)!(L_2+l'+n-4)!}{(l_1+l'+n-3)!(L_1+l'+n-3)!}\bigg)^{1/2}$$
$$\times\sum_z\frac{(-1)^z[\frac{1}{2}(l_1+l_2+L_1-L_2)+l'+n-3-z]!}{z!(l'-L_2-z)![\frac{1}{2}(l_1+l_2-L_1-L_2)-z]![\frac{1}{2}(L_1-L_2-l_1+l_2)-z]!}$$
$$\times\{(2l'+n-4-2z)!![\tfrac{1}{2}(l_1-l_2+L_1+L_2)-l'+z]!(2L_2+n-4+2z)!!\}^{-1}. \tag{2.2}$$

† Other appearances of $6j$-coefficients with quartervalued parameters are presented and discussed in appendix 4 of Ališauskas (1984).

‡ The corresponding equation (A2.3) of Ališauskas (1984) should be corrected (see the corrigendum and, for phase factor, Ališauskas (1986a).

Here the following notation of Ališauskas (1984) is used:

$$\nabla_{n[i_1\ldots i_k]}(l_1 l_2;\, L_1 L_2) = \left(\prod_{i=0}^{7} A_i\right)^{1/2} \left(\prod_{i\to[i_1\ldots i_k]} A_i\right)^{-1} \tag{2.3}$$

$$\begin{aligned} &A_0 = (l_1 + l_2 + L_1 + L_2 + 2n - 6)!! && A_4 = (l_1 + l_2 + L_1 - L_2 + n - 2)!! \\ &A_1 = (L_1 + L_2 - l_1 + l_2 + n - 4)!! && A_5 = (L_1 - L_2 - l_1 + l_2)!! \\ &A_2 = (l_1 - l_2 + L_1 + L_2 + n - 4)!! && A_6 = (l_1 - l_2 + L_1 - L_2)!! \\ &A_3 = (l_1 + l_2 - L_1 + L_2 + n - 4)!! && A_7 = (l_1 + l_2 - L_1 - L_2)!!. \end{aligned} \tag{2.4}$$

Particular cases of (2.1) (without sum) may be checked by comparison with the results of Norvaišas and Ališauskas (1974a, b), for example, when $L_1 + L_2 = l_1 + l_2$ the equivalent result follows from their corrected equation (3.5) (with a changed sign before the denominator of the penultimate term). On the other hand, when $L_1 - L_2 = l_1 - l_2$ their equation (3.7) may be used for control.

The most general case of (2.1) may be proved by means of the recurrence relation

$$\begin{aligned} &\begin{bmatrix} l_1-1 & 1 & [l_1 0]_n \\ l' & 0 & l' \end{bmatrix} \begin{bmatrix} l_1 & l_2 & [L_1 L_2]_n \\ l' & 0 & l' \end{bmatrix} \\ &\quad = \sum_{\pm} \langle l_1 - 1 \quad 1(l_1) l_2 [L_1 L_2]_n | l_1 - 1;\, l_2 1(l_2 \pm 1)[L_1 L_2]_n \rangle \\ &\quad\quad \times \begin{bmatrix} l_2 & 1 & [l_2 \pm 1, 0]_n \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} l_1 - 1 & l_2 \pm 1 & [L_1 L_2]_n \\ l' & 0 & l' \end{bmatrix}. \end{aligned} \tag{2.5}$$

The recoupling coefficients of SO_n on the RHS are found by induction and are expressed as

$$\begin{aligned} &\langle l_1 - 1 \quad 1(l_1) l_2 [L_1 L_2]_n | l_1 - 1;\, l_2 1(l_2 - 1)[L_1 L_2]_n \rangle \\ &\quad = \tfrac{1}{2}[(l_1 + l_2 - L_1 - L_2)(l_1 + l_2 - L_1 + L_2 + n - 4)(l_1 + l_2 + L_1 - L_2 + n - 2) \\ &\quad\quad \times (l_1 + l_2 + L_1 + L_2 + 2n - 6)/l_1(2l_1 + n - 4)(l_2 + n - 3)(2l_2 + n - 2)]^{1/2} \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\langle l_1 - 1 \quad 1(l_1) l_2 [L_1 L_2]_n | l_1 - 1;\, l_2 1(l_2 + 1)[L_1 L_2]_n \rangle \\ &\quad = \tfrac{1}{2}[(l_1 - l_2 + L_1 - L_2)(L_1 - L_2 - l_1 + l_2 + 2)(L_1 + L_2 + l_1 - l_2 + n - 4) \\ &\quad\quad \times (L_1 + L_2 - l_1 + l_2 + n - 2)/l_1(2l_1 + n - 4)(l_2 + 1)(2l_2 + n - 2)]^{1/2}. \end{aligned} \tag{2.7}$$

The auxiliary isofactors of $SO_n \supset SO_{n-1}$ with maximal values of parameters of SO_{n-1} irreps or with $l_2 = 1$ (needed for deriving (2.6), (2.7) and for the explicit form of (2.5)) may be found from (4.1) of Norvaišas and Ališauskas (1974b).

Equation (2.5) appears to be equivalent to the recursive formula for $6j$-coefficients of SU_2 (which may be obtained immediately from the Biedenharn–Elliott identity (Biedenharn 1953, Elliott 1953) or from (30.16) of Jucys and Bandzaitis (1977), taking into account (29.18)).

Isofactors of type (1.1) gain the phase factor $(-1)^{L_2 - L_2}$ after the permutation $l_1 \leftrightarrow l_2$, $l_1' \leftrightarrow l_2'$.

Now let us present another expression for special isofactors of $SO_n \supset SO_{n-1}$ needed for the theory of collective states:

$$\begin{bmatrix} l_1 & l_2 & [L_1L_2]_n \\ l' & l' & [L_2L_2]_{n-1} \end{bmatrix} = (-1)^{(l_1+l_2+L_1-L_2)/2+l'+n}[\tfrac{1}{2}(2L_1+n-2)(2l'+n-3)]^{1/2} \times \begin{Bmatrix} \frac{1}{2}(l_1-L_2) & \frac{1}{2}l_2+\frac{1}{4}n-1 & \frac{1}{2}L_1+\frac{1}{4}n-1 \\ \frac{1}{2}l_2+\frac{1}{4}n-1 & \frac{1}{2}(l_1+L_2+n)-2 & l'+\frac{1}{2}n-2 \end{Bmatrix}. \tag{2.8}$$

In this case the isofactor is equivalent to a recoupling coefficient of SU_2 (with quarter-valued parameters for n odd). The particular cases of (2.1) and (2.8) with $L_2=0$ are related by a symmetry relation (depending on dimensions of irreps). This symmetry relation allows us to easily check the normalisation of the general case of (2.1). Relation (1.4) allows us to prove (2.8) for arbitrary L_2.

The remaining isofactors of $SO_n \supset SO_{n-1}$ needed for constructing O_n-scalar basis functions depending on microscopic collective variables of the nucleus (as well as for the closed shell case) may be expressed in terms of some dimensional factors with the help of symmetry relations (not necessarily particular cases of (1.1)).

3. Isofactors of SO_n with two equal rows of resulting irrep and chain reflection symmetry

Expression (46) of Ališauskas and Jucys (1971) for special isofactors of $Sp_4(SO_5)$, together with the symmetry properties of Sp_4 isofactors (see Hecht 1965) and relation (2.4), allows us to obtain the following expression for special isofactors of $SO_n \supset SO_{n-1}$:

$$\begin{bmatrix} l & l & [LL]_n \\ l_1' & l_2' & [LL_2']_{n-1} \end{bmatrix} = (-1)^{l-l_1'+n}\left(\frac{(2L+n-4)(2L+n-2)(2l_1'+n-3)(2l_2'+n-3)}{2(2l+n-2)(L-L_2'+1)(L+L_2'+n-4)}\right)^{1/2} \times \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} \tag{3.1a}$$

$$= (-1)^{l-l_1'}\nabla_{n-1[-]}(l_1'l_2';\, LL_2)\left(\frac{(2L+n-4)(2L+n-2)}{(L-L_2'+1)(L+L_2'+n-4)} \times \frac{(2l_1'+n-3)(2l_2'+n-3)(l-L)!(l-l_1')!(l-l_2')!}{(2l+n-2)(l+L+n-3)!(l+l_1'+n-3)!(l+l_2'+n-3)!}\right)^{1/2} \times \sum_z \frac{(-1)^z(2l+n-3-z)!\,2^{-(l_1'+l_2'+L-L_2'+n-4)/2}}{z!(l-L-z)!(l-l_1'-z)!(l-l_2'-z)!(2l+n-3-2z)!!} \times \{[\tfrac{1}{2}(l_1'+l_2'+L-L_2')-l+z]!(l_1'+l_2'+L+L_2'-2l+n-5+2z)!!\}^{-1}. \tag{3.1b}$$

Here

$$a=\tfrac14(2l-l_1'+l_2'-L+L_2') \qquad b=\tfrac14(2l+l_1'-l_2'-L+L_2'+n-5)$$
$$c=\tfrac12(l+n-4)+\tfrac14(L-l_1'+l_2'+L_2') \qquad d=\tfrac14(2l+l_1'-l_2'+L-L_2'+n-3) \tag{3.2}$$
$$e=\tfrac14(2L+n-5) \qquad f=\tfrac12(l_1'+l_2'+n-4).$$

The parameters (3.2) of the $6j$-coefficient take standard values, from angular momentum theory, for n odd. Otherwise the quartervalued parameters appear.

Expressions (2.8) and (3.1a) may be obtained from each other by means of the analytical continuation relation

$$\begin{bmatrix} l_1 & l_2 & [L_1L_2]_n \\ l_1' & l_2' & [L_1'L_2']_{n-1} \end{bmatrix}$$
$$=(-1)^{(L_1-L_2+l_1-l_2-L_1'+L_2'-l_1'+l_2')/2}\left(\frac{(L_1-L_2+1)(L_1+L_2+n-3)}{(L_1'-L_2'+1)(L_1'+L_2'+n-4)}\right.$$
$$\left.\times\frac{(2L_1+n-2)(2L_2+n-4)(2l_1'+n-3)(2l_2'+n-3)}{(2L_1'+n-3)(2L_2'+n-5)(2l_1+n-2)(2l_2+n-2)}\right)^{1/2}$$
$$\times\begin{bmatrix} -l_1'-n+\frac52 & -l_2'-n+\frac52 & [-L_2'-n+\frac72,-L_1'-n+\frac72]_n \\ -l_1-n+\frac52 & -l_2-n+\frac52 & [-L_2-n+\frac72,-L_1-n+\frac72]_{n-1} \end{bmatrix}. \tag{3.3}$$

The symmetry of isofactors allowing the interchange of certain reflected parameters of irreps of the group and its subgroup will be called the chain reflection symmetry. Equation (3.3) may be proved with the help of the relation between isofactors of the complementary groups SO_n and Sp_4 (see (1.2) and (1.3)), the usual symmetry properties of $SO_5(Sp_4)$ isofactors (Hecht 1965) and the elements of substitution groups of parameters of irreps (Ališauskas and Jucys 1969). The phase factor is extrapolated from the isofactors considered in this paper.

Relation (3.1b) for $n=4$ allows us to write a new expression for the special SU_2 $6j$-coefficient:

$$\begin{Bmatrix} l_1 & l_2 & l_3 \\ \alpha & \alpha & \alpha \end{Bmatrix}=(-1)^{l_1+l_2+l_3}\Delta(l_1l_2l_3)(l_1+l_2+l_3+1)!$$
$$\times\left(\frac{(2\alpha-l_1)!(2\alpha-l_2)!(2\alpha-l_3)!}{(2\alpha+l_1+1)!(2\alpha+l_2+1)!(2\alpha+l_3+1)!}\right)^{1/2}$$
$$\times\sum_z\frac{(-1)^z(4\alpha+1-z)!2^{z-2\alpha}}{z!(2\alpha-l_1-z)!(2\alpha-l_2-z)!(2\alpha-l_3-z)!}$$
$$\times[(4\alpha+1-2z)!!(l_1+l_2+l_3-4\alpha+2z)!]^{-1}. \tag{3.4}$$

Here Δ is the triangle coefficient. For some values of parameters, this expression contains only half the number of terms compared with the usual expression of SU_2 $6j$-coefficients (see Jucys and Bandzaitis 1977, Biedenharn and Louck 1981). The analogue of this expression can be found in relations between special Saalschutzian ${}_4F_3(1)$ series (see (2.5.16) of Slater (1966) together with the equation (11) of § 9.2 of Varshalovich *et al* (1975)).

4. General expression for coupling coefficients of symmetric irreps

General isofactors of type (1.1) for $SO_5 \supset SO_4$ in the parametrisation of Ališauskas and Jucys (1969, 1971) may be found with the help of the following equation:

$$\begin{bmatrix} \langle\Lambda_1\Lambda_1\rangle & \langle\Lambda_2\Lambda_2\rangle & \langle K\Lambda\rangle \\ I_1 I_1 & I_2 I_2 & IJ \end{bmatrix} \left\langle \begin{matrix}\langle K\Lambda\rangle \\ IJ\end{matrix} \right\| T^{2J} \left\| \begin{matrix}\langle K\Lambda\rangle \\ K-\Lambda, 0\end{matrix} \right\rangle$$

$$= \sum_{x,I'} \frac{(2J+1)!}{x!(2J-x)!} \left\langle \begin{matrix}\langle\Lambda_1\Lambda_1\rangle \\ I_1 I_1\end{matrix} \right\| T^{x} \left\| \begin{matrix}\langle\Lambda_1\Lambda_1\rangle \\ I'I'\end{matrix} \right\rangle$$

$$\times \left\langle \begin{matrix}\langle\Lambda_2\Lambda_2\rangle \\ I_2 I_2\end{matrix} \right\| T^{2J-x} \left\| \begin{matrix}\langle\Lambda_2\Lambda_2\rangle \\ I'I'\end{matrix} \right\rangle \begin{bmatrix} \langle\Lambda_1\Lambda_1\rangle & \langle\Lambda_2\Lambda_2\rangle & \langle K\Lambda\rangle \\ I'I' & I'I' & K-\Lambda, 0 \end{bmatrix}$$

$$\times (2I_1+1)(2I_2+1)(2K-2\Lambda+1)^{1/2} \begin{Bmatrix} I' & \frac{1}{2}x & I_1 \\ I' & J-\frac{1}{2}x & I_2 \\ K-\Lambda & J & I \end{Bmatrix}$$

$$\times \begin{Bmatrix} I' & \frac{1}{2}x & I_1 \\ I' & J-\frac{1}{2}x & I_2 \\ 0 & J & J \end{Bmatrix}. \tag{4.1}$$

Here the stretched $9j$-coefficients of SU_2, the reduced matrix elements of degrees of the generators of $SO_5(Sp_4)$ (see (5) and (7) of Ališauskas and Jucys (1971)) and special isofactors of $SO_5 \supset SO_4$ of type (2.8) appear. Equation (4.1) was derived acting with the tensorial operator T^{2J}_{-+} (T_{-+} is the generator of Sp_4) on the coupled state

$$\left| \begin{matrix}\langle K\Lambda\rangle \\ K-\Lambda, K-\Lambda, 0, 0\end{matrix} \right\rangle$$

using the Wigner-Eckart theorem and the usual technique of angular momentum theory. It should be noted that the expressions for general isofactors of SU_3 and SU_n have been derived (see Ališauskas 1982, 1983) analogously.

Relations (1.4) and (4.1) allow us to write the expression for the isofactors (1.1) in the following form ($n \geq 5$):

$$\begin{bmatrix} l_1 & l_2 & [L_1 L_2]_n \\ l'_1 & l'_2 & [L'_1 L'_2]_{n-1} \end{bmatrix}$$

$$= [(L_1+L_2+n-3)(L_1-L_2+1)(2L_2+n-4)(2l'_1+n-3)(2l'_2+n-3)]^{1/2}$$

$$\times \left(\frac{(L_1+L'_2+n-4)!(L_1-L'_1)!}{2(L_1+L'_1+n-3)!(L_1-L'_2+1)!}\right)^{1/2} \sum_{x,l'} \frac{(L'_1-L'_2+1)!(2l'+n-3)^{1/2}}{x!(L'_1-L'_2-x)!}$$

$$\times (-1)^{(l'+x-l'_1)/2} R_n(l_1; l'_1 l'; x) R_n(l_2; l'_2 l'; L'_1-L'_2-x)$$

$$\times \left| \begin{Bmatrix} \frac{1}{2}x & \frac{1}{2}(L'_1-L'_2-x) & \frac{1}{2}(L'_1-L'_2) \\ \frac{1}{2}l'_2+\frac{1}{4}(n-5) & \frac{1}{2}l'_1+\frac{1}{4}(n-5) & \frac{1}{2}l'+\frac{1}{4}(n-5) \end{Bmatrix} \right|$$

$$\times \begin{Bmatrix} \frac{1}{2}l'_1+\frac{1}{4}(n-5) & \frac{1}{2}l'+\frac{1}{4}(n-5) & \frac{1}{2}x \\ \frac{1}{2}l'_2+\frac{1}{4}(n-5) & \frac{1}{2}l'+\frac{1}{4}(n-5) & \frac{1}{2}(L'_1-L'_2-x) \\ \frac{1}{2}(L'_1+L'_2+n-5) & L_2+\frac{1}{2}(n-5) & \frac{1}{2}(L'_1-L'_2) \end{Bmatrix}$$

$$\times \begin{bmatrix} l_1 & l_2 & [L_1 L_2]_n \\ l' & l' & [L_2 L_2]_{n-1} \end{bmatrix} \tag{4.2}$$

where

$$R_n(l;\ \bar{l}'l';\ x)=\{(l-l')!(l-\bar{l}')![\tfrac{1}{2}(l'-\bar{l}'+x)]![\tfrac{1}{2}(\bar{l}'-l'+x)]!(l'+\bar{l}'+x+n-3)!!$$
$$\times(l'+\bar{l}'-x+n-5)!!/(l+l'+n-3)!(l+\bar{l}'+n-3)!\}^{1/2}$$
$$\times\sum_z \frac{2^{(l'-\bar{l}')/2+z}(l+l'+n-3+z)!}{z!(l-l'-z)![\tfrac{1}{2}(\bar{l}'-l'+x)-z]!(l'-\bar{l}'+z)!(2l'+n-3+2z)!!} \qquad (4.3)$$

is the symmetrical part extracted from the analytical continuation of the reduced matrix element. For n odd, the usual stretched $9j$- and $6j$-coefficients of SU_2 appear on the RHS (the sign of the last one is included in the phase factor). For the calculation of the corresponding $9j$- and $6j$-coefficients with quartervalued parameters, equations (32.13) and (29.14) of Jucys and Bandzaitis (1977) or the corresponding equations (2) of § 10.8 and (3) of § 9.5 of Varshalovich *et al* (1975) should be applied (see also Sharp 1967).

In a similar way, the new expression for the Clebsch-Gordan coefficients of $SU_2(SO_3)$

$$\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}=(-1)^{(l_1+l_2-l+\delta')/2}\Delta(l_1l_2l)m_1!$$
$$\times\left(\frac{(2l+1)(l_1-m_1)!(l_2-m_2)!(l+m)!}{(l_1+m_1)!(l_2+m_2)!(l-m)!}\right)^{1/2}$$
$$\times\sum_{k,x}\frac{(-1)^{k-m_2+x}(2k)!(l-k)![\tfrac{1}{2}(l_1+l_2+l-\delta')+k-x]!}{(k-m_2)!(m-k)!(l+k)!(2x+\delta')!(2k-\delta'-2x)!}$$
$$\times\{[\tfrac{1}{2}(l_1+l_2-l-\delta')-x]![\tfrac{1}{2}(l_1-l_2+l-\delta')-x]!$$
$$\times[\tfrac{1}{2}(l-l_1+l_2+\delta')-k+x]!\}^{-1} \qquad (4.4)$$

is obtained. Here $m_1\geqslant 0$, $m_2\geqslant 0$, l_1 and m_1 are integers, $\delta'=0$ or 1 so that $l_1+l_2-l-\delta'$ is an even integer and Δ is the triangle coefficient. Expression (4.4) is convenient for small values of m_1 and m_2, in the same way that (4.2) is convenient for small values of $L_1'-L_2'$.

The corrected equation (A.2) of Ališauskas and Jucys (1971) (see (32.10) of Jucys and Bandzaitis (1977)†) is the simplest expression for $9j$-coefficients of SU_2, which are proportional to the isofactors of $SO_4\supset SO_3$ (see Hecht 1965) and to special isofactors of $SO_5\supset SO_4$ ($Sp_4\supset SU_2\times SU_2$) (see Ališauskas and Jucys 1969).

5. Recoupling coefficients of symmetric irreps of SO_n

Recoupling coefficients of three symmetric irreps of SO_n may be used for the calculation of the reduced matrix elements of the coupled tensor operators, similar to the Racah coefficients in angular momentum theory. It is convenient to separate more symmetric $6l$-coefficients for SO_n, analogous to $6j$-coefficients, which satisfy the usual symmetry properties of the latter. We shall denote them (for $n\geqslant 4$) as

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_{(SO_n)}=[\dim_{SO_n}(e)\dim_{SO_n}(f)]^{-1/2}\begin{bmatrix} e & d & c \\ 0 & 0 & 0 \end{bmatrix}^{-1}_{(SO_n)}$$
$$\times\sum_{l'}\begin{bmatrix} d & b & f \\ 0 & l' & l' \end{bmatrix}_{(SO_n)}\begin{bmatrix} f & a & c \\ l' & l' & 0 \end{bmatrix}_{(SO_n)}\begin{bmatrix} a & b & e \\ l' & l' & 0 \end{bmatrix}_{(SO_n)}. \qquad (5.1)$$

† The corresponding equation (3.326) of Biedenharn and Louck (1981) should be corrected.

The dimension of the symmetric irrep of SO_n is given by

$$\dim_{SO_n}(l) = (2l+n-2)(l+n-3)!/(n-2)!\,l!. \tag{5.2}$$

The perimeters of all four triangles of the $6l$-coefficient are even. Of course, the recoupling coefficients (5.1) do not form complete recoupling matrices. For $n=3$ and $n=4$ they are reducible to the well known results of angular momentum theory. For $n \geq 5$ equation (2.8) allows us to express the $6l$-coefficient as follows:

$$\begin{aligned}
\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_{(SO_n)}
&= [\tfrac{1}{8}(2c+n-2)(2d+n-2)(2e+n-2)/\dim_{SO_n}(e)\dim_{SO_n}(d)]^{1/2} \\
&\quad \times \begin{bmatrix} d & e & c \\ 0 & 0 & 0 \end{bmatrix}^{-1}_{(SO_n)} \sum_{l'} (-1)^{(c+d-e)/2+f+l'+n}(2l'+n-3) \\
&\quad \times \begin{Bmatrix} \tfrac{1}{2}b & \tfrac{1}{2}f+\tfrac{1}{4}n-1 & \tfrac{1}{2}d+\tfrac{1}{4}n-1 \\ \tfrac{1}{2}f+\tfrac{1}{4}n-1 & \tfrac{1}{2}(b+n)-2 & l'+\tfrac{1}{2}n-2 \end{Bmatrix} \\
&\quad \times \begin{Bmatrix} \tfrac{1}{2}a & \tfrac{1}{2}f+\tfrac{1}{4}n-1 & \tfrac{1}{2}c+\tfrac{1}{4}n-1 \\ \tfrac{1}{2}f+\tfrac{1}{4}n-1 & \tfrac{1}{2}(a+n)-2 & l'+\tfrac{1}{2}n-2 \end{Bmatrix} \\
&\quad \times \begin{Bmatrix} \tfrac{1}{2}a & \tfrac{1}{2}b+\tfrac{1}{4}n-1 & \tfrac{1}{2}e+\tfrac{1}{4}n-1 \\ \tfrac{1}{2}b+\tfrac{1}{4}n-1 & \tfrac{1}{2}(a+n)-2 & l'+\tfrac{1}{2}n-2 \end{Bmatrix} \left(\frac{l'!(n-3)!}{(l'+n-4)!}\right)^{1/2}.
\end{aligned} \tag{5.3}$$

The sum over l' on the RHS of (5.3) is similar to the sum which appears in the Biedenharn–Elliott identity (see (28.7) of Jucys and Bandzaitis 1977). Note in passing that, in the stretched case ($e=a+b$), this identity and the expression for the stretched $6j$-coefficients of SU_2 lead to the following expression for the $6l$-coefficient:

$$\begin{aligned}
\begin{Bmatrix} a & b & a+b \\ d & c & f \end{Bmatrix}_{(SO_n)}
&= \frac{(n-2)!\,\nabla_{n[3,7]}(c,d;\,a+b,0)}{\nabla_{n[3,7]}(c,f;\,a,0)\nabla_{n[3,7]}(d,f;\,b,0)} \\
&\quad \times \left(\frac{2^{n-3}(n-4)!!\,a!\,b!\,(2a+n-4)!!\,(2b+n-4)!!}{(a+b+n-3)!\,(2a+2b+n-2)!!}\right)^{1/2}.
\end{aligned} \tag{5.4}$$

The last two factors on the RHS of (5.3) may be expanded as

$$\begin{aligned}
&\left(\frac{l'!}{(l'+n-4)!}\right)^{1/2} \begin{Bmatrix} \tfrac{1}{2}a & \tfrac{1}{2}b+\tfrac{1}{4}n-1 & \tfrac{1}{2}e+\tfrac{1}{4}n-1 \\ \tfrac{1}{2}b+\tfrac{1}{4}n-1 & \tfrac{1}{2}(a+n)-2 & l'+\tfrac{1}{2}n-2 \end{Bmatrix} \\
&\quad = \nabla_{n[0,3,5,6]}(a,b;\,e,0)\frac{2^{e+n/2-1}}{(n-6)!!}\sum_{g\geq e}(-1)^{(g-e)/2}\frac{(g+n-3)(g-e+n-6)!!}{[\tfrac{1}{2}(g-e)]!\,(g+e+n-2)!!} \\
&\quad\quad \times [\tfrac{1}{2}(e+g)+n-4]!\left(\frac{[\tfrac{1}{2}(a-b+g)]!\,[\tfrac{1}{2}(b-a+g)]!}{[\tfrac{1}{2}(a-b+g)+n-4]!\,[\tfrac{1}{2}(b-a+g)+n-4]!}\right)^{1/2} \\
&\quad\quad \times \begin{Bmatrix} \tfrac{1}{2}a & \tfrac{1}{2}(b+n)-2 & \tfrac{1}{2}(g+n)-2 \\ \tfrac{1}{2}b & \tfrac{1}{2}(a+n)-2 & l'+\tfrac{1}{2}n-2 \end{Bmatrix}.
\end{aligned} \tag{5.5}$$

Formula (5.5) is obtained from the inverse of the expansion, following from the Biedenharn–Elliott identity applied to the product of two $6j$-coefficients:

$$\begin{Bmatrix} \tfrac{1}{2}b & \tfrac{1}{2}(b+n)-2 & l'+\tfrac{1}{2}n-2 \\ \tfrac{1}{2}a & \tfrac{1}{2}(a+n)-2 & \tfrac{1}{2}(g+n)-2 \end{Bmatrix} \begin{Bmatrix} \tfrac{1}{2}b & \tfrac{1}{2}(b+n)-2 & l'+\tfrac{1}{2}n-2 \\ \tfrac{1}{2}b+\tfrac{1}{4}n-1 & \tfrac{1}{2}b+\tfrac{1}{4}n-1 & \tfrac{1}{4}n-1 \end{Bmatrix} \tag{5.6}$$

with the help of the equation (3.8) of Ališauskas (1978b), adapted originally for biorthogonal systems of $SU_3 \supset SU_2 \times U_1$ isofactors.

The Biedenharn-Elliott identity applied once again to (5.3), substituted by (5.5), gives the following expression for the $6l$-coefficient of SO_n ($n \geqslant 5$):

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_{(SO_n)} = 2^{e+n-4}(n-2)!(n-4)\nabla_{n[0,3,5,6]}(a, b; e, 0)$$
$$\times \nabla_{n[0,1,2,3]}(c, d; e, 0) \sum_{g \geqslant e} (-1)^{(g-e)/2} \frac{(g+n-3)(g-e+n-6)!!}{[\frac{1}{2}(g-e)]!(g+e+n-2)!!}$$
$$\times [\tfrac{1}{2}(e+g)+n-4]! \left(\frac{[\frac{1}{2}(a-b+g)]![\frac{1}{2}(b-a+g)]!}{[\frac{1}{2}(a-b+g)+n-4]![\frac{1}{2}(b-a+g)+n-4]!} \right)^{1/2}$$
$$\times \begin{Bmatrix} \frac{1}{2}a & \frac{1}{2}(b+n)-2 & \frac{1}{2}(g+n)-2 \\ \frac{1}{2}d+\frac{1}{4}n-1 & \frac{1}{2}c+\frac{1}{4}n-1 & \frac{1}{2}f+\frac{1}{4}n-1 \end{Bmatrix}$$
$$\times \begin{Bmatrix} \frac{1}{2}(a+n)-2 & \frac{1}{2}b & \frac{1}{2}(g+n)-2 \\ \frac{1}{2}d+\frac{1}{4}n-1 & \frac{1}{2}c+\frac{1}{4}n-1 & \frac{1}{2}f+\frac{1}{4}n-1 \end{Bmatrix}. \tag{5.7}$$

Since (5.7) is rather non-symmetric it is convenient to use the symmetry property of $6l$-coefficients which allows us to replace $a+b-e$ by the smallest one from twelve corresponding linear combinations of the parameters.

6. Concluding remarks

The explicit expressions obtained demonstrate some new non-trivial applications of the angular momentum theory for compact groups of a higher rank. However, the appearance of $6j$-coefficients of SU_2 in these cases still remains unexplained. The crucial result (equations (2.1) and (2.8)) was extrapolated from the $n=3$ and $n=4$ cases and later the proof of it was found. In the context of (4.1) and (4.2) some vague analogy is made with the $SU_3 \supset U_2$ isofactors, which for extremal values of parameters are proportional to $6j$-coefficients of SU_2 (Ališauskas 1978b, 1982).

References

Ališauskas S J 1974 *Liet. Fiz. Rink.* **14** 861-73 (*Sov. Phys.-Coll.* (*Lit. Fiz. Sb.*) **14** no 6, 1)

—— 1976 *Liet. Fiz. Rink.* **16** 359-70 (*Sov. Phys.-Coll.* **16** no 3, 12-9)

—— 1978a *Liet. Fiz. Rink.* **18** 567-91 (*Sov. Phys.-Coll.* **18** no 5, 1-20)

—— 1978b *Liet. Fiz. Rink.* **18** 701-13 (*Sov. Phys.-Coll.* **18** no 6, 6-16)

—— 1982 *Liet. Fiz. Rink.* **22** no 2, 13-9 (*Sov. Phys.-Coll.* **22** no 2, 9-14)

—— 1983 *Part. Nucl.* **14** 1336-79 (*Sov. J. Part. Nucl.* **14** 563-82)

—— 1984 *J. Phys. A: Math. Gen.* **17** 2899-926 (corrigendum 1985 *J. Phys. A: Math. Gen.* **18** 737)

—— 1986a *J. Phys. A: Math. Gen.* **19** 1761-73

—— 1986b *Liet. Fiz. Rink.* **26** no 5 646-9

Ališauskas S J and Jucys A P 1969 *J. Math. Phys.* **10** 2227-33

—— 1971 *J. Math. Phys.* **12** 594-605 (erratum 1972 *J. Math. Phys.* **13** 575)

Ališauskas S J and Norvaišas E Z 1975 *Liet. Fiz. Rink.* **15** 317-28 (*Sov. Phys.-Coll.* **15** no 3, 1-8)

Biedenharn L C 1953 *J. Math. Phys.* **31** 287-93

Biedenharn L C and Louck J D 1981 *Angular Momentum in Quantum Physics. Encyclopedia of Mathematics and its Applications* vol 8 (Reading, MA: Addison-Wesley)

Castilho-Alcarás J A and Vanagas V V 1984 *Rev. Brasil Fiz.* **14** 121-31

—— 1985 *J. Math. Phys.* submitted

Elliott J P 1953 *Proc. R. Soc.* A **218** 345-70

Gavrilik A M 1973 *Kiev preprint* ITP-73-104R

Hecht K T 1965 *Nucl. Phys.* **63** 177-213

Jucys A P and Bandzaitis A A 1977 *Theory of Angular Momentum in Quantum Mechanics* (Vilnius: Mokslas) 2nd edn

Kildyushov M S and Kuznetzov G I 1973 *Preprint, Institute of Atomic Energy, Moscow*

Norvaišas E Z and Ališauskas S J 1974a *Liet. Fiz. Rink.* **14** 443-52 (*Sov. Phys.-Coll.* **14** no 3, 17-22)

—— 1974b *Liet. Fiz. Rink.* **14** 715-25 (*Sov. Phys.-Coll.* **14** no 5, 5-11)

Sharp R T 1967 *Nucl. Phys.* A **95** 222-8

Slater L J 1966 *Generalized Hypergeometric Functions* (Cambridge: Cambridge University Press)

Vanagas V 1977 *The Microscopic Nuclear Theory* (Toronto: University of Toronto Press)

—— 1980 *Part. Nucl.* (*Sov. J. Part. Nucl.*) **11** 454-514

Vanagas V V and Kalinauskas R K 1974 *Liet. Fiz. Rink.* **14** 549-60 (*Sov. Phys.-Coll.* **14** no 4, 3-10)

Vanagas V V and Katkevičius O D 1983 *Liet. Fiz. Rink.* **23** no 2, 3-13 (*Sov. Phys.-Coll.* **23** no 2, 1-9)

Varshalovich D A, Moskalev A N and Khersonsky V K 1975 *Quantum Theory of Angular Momentum* (Leningrad: Nauka)